Schwarz-Christoffel Transformation and Applications in Applied Probability

Yvik C. Swan and F. Thomas Bruss

Université Libre de Bruxelles

Summary

Problems involving stochastic processes frequently involve the computation of hitting probabilities, and one often has to approximate the given process by a Brownian motion. But, in order to obtain explicit answers, it is sometimes necessary to map the Brownian motion and the region in which it is defined, in such a way that the transformed process is again a Brownian motion. The desired exit probabilities can then be found by symmetry arguments. For planar Brownian motion, the Schwarz-Christoffel transformation is such a mapping. The goal of this paper is to provide an organized summary of the relevant theory and a step-by-step guide to finding the explicit form of the transformation. We are mainly concerned with exit probability problems. We also draw attention to software developped by Driscoll and Trefethen which we found very helpful. Among the new problems we solve is the three players' ruin problem with capital constraints.

Keywords: Complex functions, Riemann's Theorem, conformal mapping, Lévy's theorem, Brownian motion, Moebius transform, Schwarz-Christoffel transformation, Ruin problem with constraints.

AMS 2000 subject classification: 60

1 Introduction

Exit probabilities or expected hitting times for stochastic processes are recurrent themes in probability, but their explicit computation can be complicated. Approximating a given stochastic process by a simpler one is therefore a natural approach, and Brownian motions play an important role in such a replacement. However, for processes of two or higher dimensions, the mathematical justification for replacement by a Brownian motion of the same dimension is usually not sufficient to find the explicit solutions of hitting time problems. So, for instance, for a planar Brownian motion, the probability of hitting a certain region R in a subregion S of the plane, conditioned on being in S does not depend only on the shape of R but also on that of S. The same is true in higher dimensions.

To start with a specific example, suppose we have a Brownian motion $(W_t)_{t\geq o}$ without drift, starting at the origin ($W_0 = (0,0)$). Let D_r be the disc centered in $(0,0)$ with radius r, and let L be a fixed arc segment of length l on the circumference of the circle. The probability that the first exit of (W_t) will occur through L is just the angular measure of L with respect to the origin , that is $l/(2\pi r) = \phi/(2\pi)$.

Clearly, in this specific case, the symmetry argument is not only sufficient for the answer to be intuitively clear, but also to prove its correctness. Moving the starting point W_0 away from the origin changes the situation completely. If the angular measure intervenes at all in the answer, it is clearly not independent of r. More generally the angular measure of L is bound to play a subordinate role in the answer, depending on the shape of the region S.

A powerful tool is Lévy's theorem (see Lévy(1965), Theorem 56.1 page 254).

> **Lévy's Theorem**: The intrinsic properties of a Brownian motion remain invariant under conformal transformations.

Other versions of this theorem, and its applications can be found in Bass(1995) or Durrett(1984).
As a corollary one has the following (Theorem 56.2 p 255 of Lévy(1965)):

> If a Brownian motion initially occupies a position A interior to a contour Γ, the probability that it reaches Γ for the first time through an arc Γ' is proportionnal to the harmonic measure of Γ' seen from A.

Lévy explains how one can construct the harmonic measure of an arc, but we shall not require this here. What is of interest to us in this theorem is the following rule. To compute the exit probability of a Brownian motion through an arc Γ' of a given domain Γ starting at a point A in this domain, it suffices to construct a conformal transformation from Γ to the unit disk which maps A into its center, and compute the image of Γ' through the conformal transformation. Then, by the same symmetry argument as above, the desired probability will simply be the length of the image arc divided by 2π.

Let us now recall some elements of the theory of complex functions. This material has been widely covered in many good books (see for example Churchill et al.(1974), Hildebrand(1963) or Rudin(1966)). For the whole of this paper we shall be working with the extended complex plane, i.e. the complex plane in which we include the point at infinity (denoted by ∞). All other points $z \in \mathbb{C}$ are finite.

First of all, a function will be said to be *analytic* in a region R of the complex plane if it has a finite derivative at each point in R, and is single-valued in R. *Cauchy's theorem* (see e.g. Rudin(1966) page 207) states that the line integral of an analytic function over any closed piecewise differentiable curve vanishes. This means that the integral of an analytic function from a point $a \in \mathbb{C}$ to a point $b \in \mathbb{C}$ is independent of the path chosen. We also recall that a mapping $f : \mathbb{C} \to \mathbb{C}$ is said to be conformal if it preserves the angle between two intersecting differentiable arcs. One can see that any mapping which is analytic over a domain, and has derivative $f'(z_0) \neq 0$ at every point z_0 in the domain is a conformal transformation of that domain.

An important example of conformal transformations is that of the linear fractional transformation, also known as the *Moebius transformation*. This is of the form

$$w = \frac{az+b}{cz+d} \quad (ad - bc \neq 0).$$

Such a transformation is one to one from the extended plane to itself, and is uniquely determined up to the choice of three distinct points and their three distinct images (see Bieberbach(1953) pages 24, 25, 26). A special subfamily is that of the linear fractional transformation which maps the upper half plane Im$z \geq 0$ onto the unit disk $|w| \leq 0$. It is of the form

$$w = e^{i\alpha}\frac{z - z_0}{z - \bar{z}_0}, \tag{1}$$

where the point $z_0 \in$ Im$z > 0$ is sent into the center of the disk.

One of the fundamental building blocks of this theory is the following theorem, of which there are many versions (see for example Rudin(1966) page 273 or Bieberbach(1953) page 128).

> **Riemann's Theorem:** There exists a conformal one to one transformation from any simple and simply connected region Ω in the plane (other than the plane itself) to the unit disk. It is uniquely determined up to the choice of three points in Ω and their images.

Since the inverse of a one-to-one conformal transformation is another conformal transformation, the theorem also implies that any two simply connected regions of the plane are conformally equivalent, i.e. can be mapped onto one another by a one to one conformal transformation.

Conformal transformations have a large number of applications in physics, fluid dynamics, partial differential equations, electrostatics, etc., since many properties remain invariant after such transformations. One can therefore transpose a problem in a difficult domain into a similar problem in an easier or more appropriate domain.

However Riemann's theorem is of no help when one wishes to construct a conformal transformation explicitely mapping a given domain into another. In the next section we shall outline a systematic method for the construction of a transformation mapping a general polygon into a circle. This turns out to take a nice form, known as the *Schwarz-Christoffel transformation*.

2 The Schwarz-Christoffel transformation

This is a family of conformal transformations of a given form (see equation (**??**) below) which map a canonical domain (unit disk, upper half plane...) conformally onto the interior of a polygon. Figures **??** and **??** illustrate the Schwarz-Christoffel transformation of the upper half plane and the unit disk (respectively) into a triangle. These mappings were named after Hermann A. Schwarz (1843-1921) and Elwin B. Christoffel (1829-1900), who discovered them independently.

(PUT FIGURE **??** around here)

2.1 Construction

Consider a smooth directed arc $C = z(t)$ in the complex z-plane. Let v denote the unit vector tangent to C at a point $z_0 := z(t_0)$. Let τ denote the unit vector tangent to the image Γ of C in the complex w-plane under a transformation $w = f(z)$ at the corresponding point $w_0 := f(z_0)$. Suppose that the transformation f is analytic at z_0 and $f'(z_o) \neq 0$ (see figure **??**).

(PUT FIGURE **??** around here)

Then, since $\log(w'(t)) = \log(f'(z(t))) + \log(z'(t))$, we have: $\arg[\tau] = \arg[f'(z_0)] + \arg[v]$. In particular if C is a positively directed segment of the x-axis, we have $v = 1$ and $\arg[v] = 0$ at every $z_0 = x$ on C, i.e.

$$\arg[\tau] = \arg[f'(z_0)]. \tag{2}$$

If $f(z)$ has constant argument along that segment, it therefore follows that $\arg[\tau]$ is constant and that the image Γ of C is also a segment of a straight line.

We now exhibit a general formula for a conformal mapping from the upper half plane onto any given closed convex polygon P, adding the further restriction that it maps the x-axis onto the border of the polygon defined by the vertices $w_1, w_2, \ldots, w_n$ (counted in a counter clockwise direction).

Let $-\infty < x_1 < x_2 < \ldots < x_{n-1} < \infty$ denote the $n-1$ real points which are to be mapped (in that order) onto the first $n-1$-vertices of P, and take w_n as the image of the point at infinity ($w_n := f(x_n)$) with x_n chosen as the point at infinity. The x_i's will be called the *prevertices* of the polygon (see figure **??**).

(PUT FIGURE **??** around here)

From the first argument it is clear that what we need is a transformation $w = f(z)$ whose derivative has constant argument along each of the segments $]x_k, x_{k+1}[$, and such that $\arg[f'(z)]$ changes value abruptly at each point $z = x_k$. Therefore, let us consider the properties of functions whose derivatives are given by

$$f'(z) = A(z - x_1)^{-k_1}(z - x_2)^{-k_2} \ldots (z - x_{n-1})^{-k_{n-1}} \tag{3}$$

with $A \in \mathbb{C}$ and the k_i's $\in \mathbb{R}$. Such a function obviously satisfies:

$$\arg[f'(z)] = \arg[A] - k_1 \arg[z - x_1] - \ldots - k_{n-1} \arg[z - x_{n-1}] \tag{4}$$

Let us take a point moving along the real axis $z = x \in \mathbb{R}$ in a positive direction, starting at $-\infty$ up to x_1. Since $z < x_i$ for all $i = 1 \ldots n-1$, all the summands in equation (**??**) (except $\arg[A]$) are constant and equal to π. But as the point passes x_1, the term $\arg[z - x_1]$ reduces to zero, since $z - x_1 > 0 \in R$. The other terms remain constant and equal to π. From equation (**??**) this implies that $\arg[f'(z)]$ increases abruptly at the prevertex x_1 by the angle $k_1\pi$ but remains constant from there up to x_2 (see figure **??**). As the moving point passes x_2 the argument of the derivative of our function increases again by the angle $k_2\pi$.

$$\begin{array}{ll} -\infty \to x_1 & \arg[f'(z)] = \arg[A] - (k_1 + k_2 + k_3 + \ldots + k_{n-1})\pi, \\ x_1 \to x_2 & \arg[f'(z)] = \arg[A] - (0 + k_2 + k_3 + \ldots + k_{n-1})\pi, \\ x_2 \to x_3 & \arg[f'(z)] = \arg[A] - (0 + 0 + k_3 + \ldots + k_{n-1})\pi. \end{array}$$

We can repeat this argument at each prevertex x_i. From equation (**??**) it follows that the image of each segment $]x_i, x_{i+1}[$ is a line segment $]w_i, w_{i+1}[$ in the w-plane, and that the exterior angle at each change of direction z_i is given by $k_i\pi$. We note that from x_{n-1} to x_n all but one the summands in equation (**??**) vanish, which implies that $\arg[f'(z)] = \arg[A]$ along that segment. Setting $k_n\pi = 2\pi - (k_1 + \ldots + k_{n-1})\pi$ yields the fact that from $-\infty$ to x_1, $\arg[f'(z)] = \arg[A] + k_n\pi$. Therefore it suffices to set $k_i > 0 \;\forall i = \ldots n$ to ensure that the image, through a function f whose derivative is given by equation (**??**), of a point moving in positive direction along the x-axis in the w-plane is a positively oriented closed convex polygon with vertices $w_1, \ldots, w_n$ and exterior angles $k_1\pi, \ldots, k_n\pi$.

(PUT FIGURE **??** around here)

It should be noted that if $k_1 + \ldots + k_{n-1} = 2$, then there is no vertex w_n since there is no change of direction at that point, and the image polygon has therefore $n-1$ sides. This implies that we can lift the restriction that one of the prevertices is the point at infinity, if we wish to do so. However, for practical reasons, this is rarely done.

All this is of course not conclusive, since one still has to prove that such a mapping exists, is conformal and one to one. It has this property everywhere on the upper half plane except at the x_i's where it is no longer conformal. This can be shown in various ways, and the two most common proofs are given in the books by Nehari(1952) and Churchill et al.(1974). We refer the interested reader to these. Taking these results for granted, we may continue.

(PUT FIGURE **??** around here)

There is a one to one correspondence between points on the x-axis and points on P. If z is some point interior to the upper half plane, and x_0 some point on the real axis (different from the x_i's), the since f is conformal throughout the upper half plane, the angle formed by the vector joining x_0 and z must be preserved. Thus, the image of interior points of the upper half plane lies to the left of the polygon taken counterclockwise (see figure **??**).

To make things more precise, we state:

> **The Schwarz-Christoffel Formula:** Given a closed convex polygon P with vertices $w_1, \ldots, w_n$ and exterior angles $k_1\pi, \ldots, k_n\pi$ (taken in counterclockwise order), there exist $n-1$ real constants $x_1 < \ldots < x_{n-1}$ and two complex constants A, B such that the mapping
>
> $$f(z) = A \int_{z_0}^{z} (s - x_1)^{-k_1} \ldots (s - x_{n-1})^{-k_{n-1}} ds + B \tag{5}$$
>
> is a conformal transformation of the upper half plane into the interior of P, which maps the real axis onto P, each x_i to the corresponding w_i and the point at infinity to w_n. This mapping is continuous throughout the upper half plane $y \geq 0$ and conformal except at the prevertices.

Lifting the restriction $x_n = \infty$ clearly adds one term to the integrand of (**??**), of the form $(s - x_n)^{-k_n}$. It is also clear that we do not have to restrict the domain to the upper half plane. For example, we can combine the inverse of a Moebius transformation in equation (**??**) with a Schwarz-Christoffel transformation to obtain a conformal transformation from the unit disk to a polygon (see Nehari(1952) p 193, Hildebrand(1963) p 577, or section **??**).

2.2 Choice of constants

Let P be a polygon with vertices $w_1, \ldots, w_n$ and corresponding exterior angles $k_1, \ldots, k_n$ (we know the k_i's in equation (**??**) are predetermined). In order for f to map the entire x-axis onto P, we must have the following n equalities

$$f(x_1) = w_1, \ldots, f(x_n) = w_n$$

However these can be simplified: writing $f = AF + B$, one sees that the complex constant $A = |A|e^{i\arg[A]}$ comprises an arbitrary magnification factor $|A|$ and a rotation by the angle $\arg[A]$. In the same way, the constant $B = b_0 + ib_1$ represents an arbitrary translation without distortion through the vector $b_0 + ib_1$. Therefore, using

$$F(z) = \int_{z_0}^{z} (s - x_1)^{-k_1} \ldots (s - x_{n-1})^{-k_{n-1}} ds, \tag{6}$$

if we determine the x_i's in such a way that F maps the x-axis onto a polygon P' similar to P, we can then choose a magnification, rotation and translation through A and B to map P' onto P. Thus these two constants are also predetermined by the position and orientation of the polygon P, and we are left to choose the x_i's to ensure that the image through F of the x-axis is a polygon similar to P.

Knowing that there are an infinite number of ways to map the upper half plane onto itself, we anticipate some freedom in their choice. This is indeed the case. The image polygon P' through equation (**??**) has the same exterior angles as P. Therefore, it suffices to make sure that the $n-2$ connected sides of P' have a common ratio to the corresponding sides of P. This yields $n-3$ equations in the $n-1$ unknowns x_i. Therefore, two of these, or two relations between them can be chosen arbitrarily (provided of course that the correponding system has $n-3$ real solutions). This means we have two degrees of liberty. This is exactly what had been predicted by Riemann's theorem. It should be noted that when the condition $x_n = \infty$ is removed, we have in fact three degrees of freedom.

The remaining $n-3$ prevertices are then uniquely determined and can be obtained by solving a system of nonlinear equations. This is non trivial and is known as the *Schwarz-Christoffel parameter problem.* See for example Howell (1990) for an overview of the problem and its inherent difficulty.

3 Applications of the Schwarz-Christoffel transformation.

3.1 Exit probability for an infinite strip

Let us take a planar Brownian motion starting at some point within an infinite strip which we suppose (without loss of generality) to be of width π. We are looking for an explicit link between the distance from the initial point $w_0 = x_0 + iy_0$ to the borders and the exit probability of the Brownian motion through one of the borders. This example could of course be solved without the use of the Schwarz-Christoffel transformation , but serves here to illustrate the method.

By viewing the strip as the limiting form of a rhombus with vertices $w_1 = i\pi, w_2, w_3 = 0, w_4$ and corresponding exterior angles $k_1\pi = 0 = k_3\pi, k_2\pi = \pi = k_4\pi$, we can use the Schwarz-Christoffel transformation to determine a conformal transformation from the upper half plane to the infinite strip. Choose the prevertices $x_2 = 0, x_3 = 1, x_4 = \infty$. (Since we have only two degrees of freedom, x_1 is left to be determined). We need a transformation f such that $f(x_1) = i\pi$, $f(x_2) = w_2$, $f(x_3) = 0$, and $f(x_4) = w_4$. This mapping has the derivative

$$\frac{df}{dz} = A(z - x_1)^0 z^{-1}(z-1)^0 = \frac{A}{z}$$

so that $f(z) = A\log(z) + B$.

In order to determine A, B and x_1, we must use the conditions on the prevertices, which yield $A = 1$, $B = 0$ and $x_1 = -1$. Therefore the mapping we are looking for is given by $w = F(z) = \log(z)$. Its inverse is therefore a conformal one to one transformation of the infinite strip into the upper half plane. Let us combine it with a Moebius transformation (see equation (**??**)) which maps $z_0 = F^{-1}(w_0) = e^{w_0}$ into the center of the unit disk. We have then determined a one to one conformal transformation of the infinite strip into the unit disk which maps w_0 into its center:

$$F_{w_0}(w) = i\left(\frac{e^w - e^{w_0}}{e^w - \overline{e^{w_0}}}\right) \tag{7}$$

The prevertices of our polygon are $x_1 = -1, x_2 = 0, x_3 = 1$ and $x_4 = \infty$. Using equation (**??**), we can compute

$$\begin{aligned} F(w_2) = &\ \cos(\tfrac{\pi}{2} + 2y_0) + i\sin(\tfrac{\pi}{2} + 2y_0) \\ F(w_4) = &\ i \end{aligned}$$

Therefore the probability that the Brownian motion will exit the strip through the upper edge is given by y_o/π, independently of x_0, as expected.

3.2 Application of the Schwarz-Christoffel transformation to the three players' ruin problem

We are now going to investigate the following ruin problem: three gamblers with initial assets a, b, c play a sequence of fair games, such that in each game one player receives one unit from each of the other two players, until one of them is ruined. We are looking for the probability that a particular player is ruined first (for an extensive study of different ruin problems see for example Asmussen(2000)).

The total assets available throughout the game remains constant ($a + b + c = \Sigma$), so we can view this as a random walk in the plane on the grid of integer coordinates of the equilateral triangle $X + Y + Z = \Sigma,\ \ X, Y, Z \geq 0$.

Letting the number of games tend to infinity, we model this as a Brownian motion in the same plane, and are therefore looking for the probability that a Brownian motion starting at (a, b, c) exits the triangle, say, along the edge $z = 0$. As one can see, this is exactly the kind of probability problem to which the preceding sections provide an answer. For a look at the discrete case of this problem and other related problems, see Bruss et al.(2003). The idea for this model is due to Ferguson (see Ferguson(1995)). Our approach, however, is more detailed.

3.3 Brownian motion on a triangle.

After a change in coordinates, this problem can clearly be transformed into the same problem on the equilateral triangle of vertices $\Delta := [-1\ \ 1\ \ i\sqrt(3)]$ in the upper half complex plane (H^+). We can then use Lévy's theorem to derive the following procedure:

1. Construct a conformal transformation F_{p_0} from the triangle to the unit disk which maps the starting point $p_0 = (a, b, c)$ into the center of the disk.

2. Compute the images of each of the three summits of the triangle through this mapping.

3. Compute the desired probabilities. For example, the probability that the third player is ruined first is given by the length of the image arc joining $F_{p_0}(-1)$ to $F_{p_0}(1)$, divided by 2π.

First of all we choose a Schwarz-Christoffel transformation from the upper half plane H^+ into Δ, say F. This transformation being one to one, we can take its inverse to get a mapping from Δ into H^+, which maps the starting point w_0 into $F^{-1}(w_0) := z_0$.

We can then use a Moebius transformation to map H^+ into the unit disk, with z_0 mapped into the center of this disk. One such transformation would be:

$$M_{z_0}(z) = i\left(\frac{z - z_0}{z - \overline{z_0}}\right).$$

The mapping F_{p_0} will be:

$$F_{p_0} = M_{F^{-1}(w_0)} \circ F^{-1}. \tag{8}$$

If the exterior angles of a triangle $< w_1, w_2, w_3 >$ are denoted respectively by $k_1\pi, k_2\pi, k_3\pi$ the general form of the Schwarz-Christoffel transformation from the upper half plane to the triangle such that $w_1 = F(x_1), w_2 = F(x_2), w_3 = F(\infty)$ is

$$w = A \int_{z_0}^{z} (s - x_1)^{-k_1} (s - x_2)^{-k_2} ds + B, \tag{9}$$

with A, B complex constants, and $z_0 \neq x_1, x_2, \infty \in \{\mathrm{Im} z \geq 0\}$. The integrals in (**??**) do not represent elementary functions unless the triangle is degenerate. In the case which is of interest to us, the triangle is equilateral ($k_1 = k_2 = \frac{2}{3}$). If we choose (arbitrarily) $x_1 = -x_2 = 1$ and $z_0 = 1$, the mapping then becomes:

$$w = F(z) = A \int_{1}^{z} (s - 1)^{-\frac{2}{3}} (s + 1)^{-\frac{2}{3}} ds + B. \tag{10}$$

Depending on the values of A and B this maps the upper half plane onto the interior of any equilateral triangle in the complex plane. In order to determine the values of A and B we use the conditions: $F(-1) = -1 \quad$ and $\quad F(1) = 1$. It will then follow that $F(\infty) = i\sqrt(3)$.

The first condition trivially implies $B = 1$.

Let us compute $\int_1^{-1} (s^2 - 1)^{-\frac{2}{3}} ds = -\int_{-1}^{1} (s^2 - 1)^{-\frac{2}{3}} ds$. If we choose a path of integration $z = t$ along the real axis in the positive sense, by writing

$$\begin{aligned} s - 1 &= |s - 1| e^{i\phi} \\ s + 1 &= |s + 1| e^{i\psi}. \end{aligned}$$

We see that the argument $\phi + \psi$ remains constant throughout integration from -1 to 1 since $s + 1$ stays positive with zero argument, and $s - 1$ has constant argument π. Therefore equation (**??**) yields

$$-A \int_{-1}^{1} (s - 1)^{-2/3} (s + 1)^{-2/3} ds - 1 = 1,$$

or, equivalently

$$-A(-e^{-2\pi i/3}) \int_{-1}^{1} (t^2 - 1)^{-2/3} dt = 2. \tag{11}$$

After manipulation of the definition of the *beta function*

$$B(\alpha, \beta) := \int_0^1 t^{\alpha - 1} (1 - t)^{\beta - 1} dt,$$

one notices that the integral in (**??**) is the beta function evaluated for the parameters $(\frac{1}{2}, \frac{1}{3})$, which implies that

$$A = \frac{2e^{2\pi i/3}}{B(\frac{1}{2}, \frac{1}{3})}.$$

Finally, we have

$$1 = \frac{B(\frac{1}{2},\frac{1}{3})}{B(\frac{1}{2},\frac{1}{3})} = \frac{2e^{2\pi i/3}}{\beta(\frac{1}{2},\frac{1}{3})}\int_0^1 (s^2-1)^{-2/3}ds.$$

We therefore obtain the following neat formula for a conformal transformation of the upper half plane into the equilateral triangle with vertices ± 1 and $i\sqrt{3}$,

$$w = F(z) = \frac{2e^{2\pi i/3}}{B(\frac{1}{2},\frac{1}{3})}\int_0^z (s^2-1)^{-2/3}ds. \tag{12}$$

It should be noted that the integration in (**??**) is that of a complex function over a path going from 0 to z in the complex plane, which is not equivalent to the integration of a real-valued function over the real line. Furthermore, the inversion of the function given in equation (**??**) is in general not expressible in terms of elementary functions. As an example of a case in which this inversion is possible manually, we choose the upper limit of the integrand to be purely imaginary, $z = iy$. The mapping then becomes (see Ferguson(1995))

$$w = i\sqrt{3}B(1/2, 1/6, \frac{y^2}{1+y^2})B(1/2, 1/6)$$

where $B(\alpha,\beta)$ is as above and $B(x,\alpha,\beta)$ is the *incomplete beta function*

$$B(x,\alpha,\beta) := \frac{1}{B(\alpha,\beta)}\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt.$$

It should be pointed out that there is no such simple formula for general $z \in \mathbb{C}$.

3.4 Tools and computations

A toolbox for MATLAB is available (as free software distributed in ZIP archives) on the internet, at the following address:

http://www.math.udel.edu/ driscoll/software/SC/.

This toolbox was designed by Driscoll (see Driscoll(1996)) as an extension of a FORTRAN package developped by Trefethen (see Howell and Trefethen(1990)) in the early 1980's. It requires no programming by the user. Its user's guide is very clearly written and makes it relatively easy and efficient to use. Thanks to this we can conveniently compute various exit probabilities.

Once the vertices of the polygon are entered, the program computes (for a chosen canonical domain such as the upper half plane, the unit disk or others) the coordinates of the *prevertices* and, when the canonical domain is chosen to be the unit disk, their arguments. It also reports the image of the origin, known as the *conformal center* of the mapping. We have a-priori no choice of this image, but, through a Moebius transformation, we can obtain a new conformal transformation of the unit disk onto the polygon, with any interior point as conformal center.

We therefore obtain exactly the inverse of the desired mapping F_{p_0}. The arguments of the prevertices are all the numerical data needed to compute the required probabilities. They will be given by $(\theta_1 - \theta_{-1})/2$ where θ_p represents the argument of the prevertex of the summit p.

(PUT FIGURE **??** AROUND HERE)

Figure **??** shows the image after the Schwarz-Christoffel transformation of the unit disk with conformal center $i\sqrt{3}/3$ of ten evenly spaced circles centered at the origin, and ten evenly spaced radii. All the intersections are orthogonal.

As an example, we choose

```
h= center(f, i*sqrt(3)/3),
```

i.e. the conformal center is at $i\sqrt{3}/3$. This means that all the players have the same initial amount of money to begin with. We show a typical output of the toolbox, for the computation of the mapping h above.

```
      vertex              alpha          prevertex              arg/pi
 --------------------------------------------------------------------------
-1.00000 + 0.00000i     0.33333    -0.50000 + 0.86603i       0.6666666666
 1.00000 + 0.00000i     0.33333    -0.50000 - 0.86603i       1.3333333333
 0.00000 + 1.73205i     0.33333     1.00000 + 0.00000i       2.0000000000
```

As expected, the probability for the Brownian motion to exit the triangle by the lower edge is exactly one third. We give a list of various probabilities, depending on the starting point:

Δ	a, b, c	p
$i\sqrt{3}/3$	$a = b = c$	0.33333
$i\sqrt{3}/5$	$a = b = 2c$	0.5617
$-1/8 + i5\sqrt{3}/8$	$a = \Sigma/8, b = \Sigma/4$	0.0534
$1/8 + 5\sqrt{3}/8$	$a = \Sigma/4, b = \Sigma/8$	0.0534

where a, b, c represent the assets of each player, Δ the point in the complex plane associated with this triplet, and p the probability that player 3 is ruined first.

The three player ruin problem with capital constraints

Another interesting problem is that of computing the probability of, say, player 3 being ruined, while players 1 and 2 still have certain defined assets. With our preparation this is now straightforward. It suffices to enter the triangle as a pentagon with four of its vertices aligned. For example,

```
p = polygon([-1 -0.5 0.5 1 i*sqrt(3)])
```

describes the case in which we are looking for the probability of player 3 being ruined first, while the other two players have assets ranging between ($\frac{\Sigma}{4}$ and $\frac{3\Sigma}{4}$), and ($\frac{3\Sigma}{4}$ and $\frac{\Sigma}{4}$), respectively. It is understood, of course, that the constraints imposed on the ranges of the assets of the two remaining players at the instant of ruin are such that the sum of remaining non-negative assets is again Σ.

We give below a list of various probabilities, depending on the starting point and the constraints on the assets. In these examples we choose equal initial capital for all players, which means that the Brownian motion starts at the center.

Δ	constraints	p
$i\sqrt{(3)}/3$	$(-0.5, 0.5)$	0.2589
$i\sqrt{(3)}/3$	$(-1, -0.9)$	$1.8999.10^{-4}$
$i\sqrt{(3)}/3$	$(-0.9, -0.1)$	0.1284
$i\sqrt{(3)}/3$	$(-0.1, 1)$	0.2048
$i\sqrt{(3)}/3$	$(-0.6, 0.8)$	0.3197

It is clear that if one modifies the problem slightly so as to look for the probability of exit through a union of disjoint intervals, the probability is the sum of the probabilities of exit through each of the intervals.

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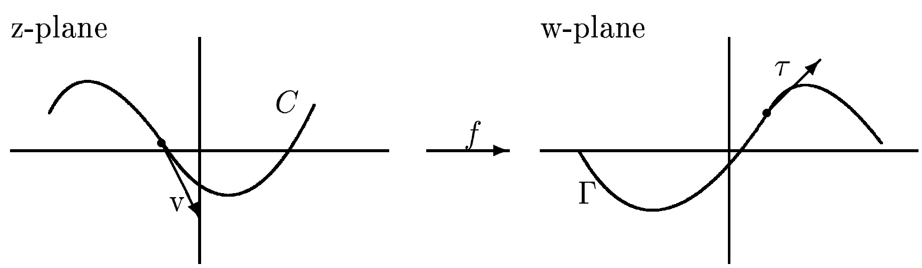
Département de Mathématique, CP 210, Boulevard du Triomphe, B-1050 Bruxelles, Belgium. e-mail: yvswan@ulb.ac.be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C
v
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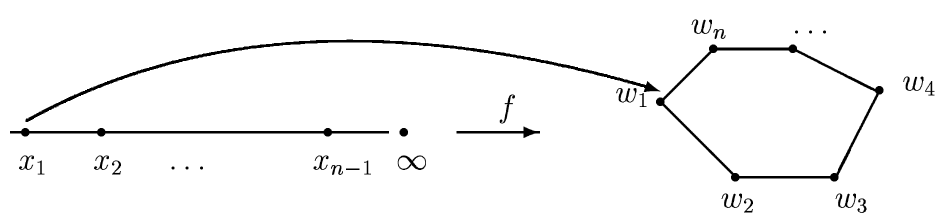
Γ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$\dots$
w_1
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f
x_1
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$\dots$
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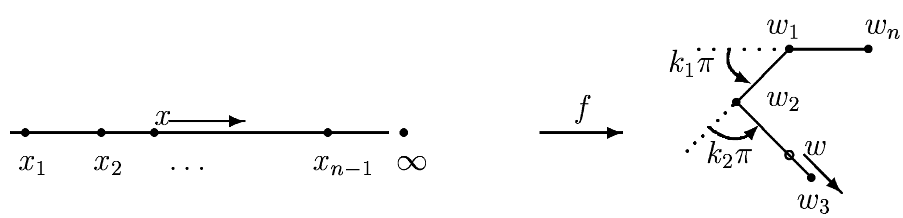
w_2
w_3



x
x_1
x_2
$\dots$
x_{n-1}
∞
f
w_1
w_n
$k_1\pi$
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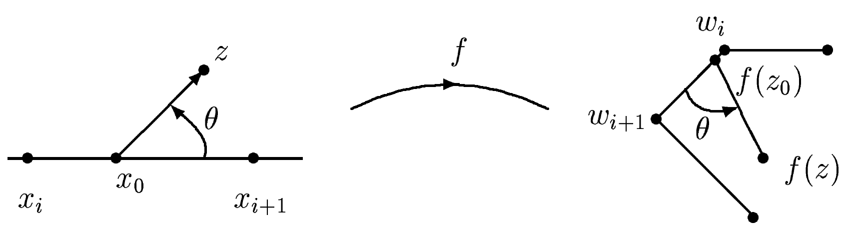
w
w_3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θ
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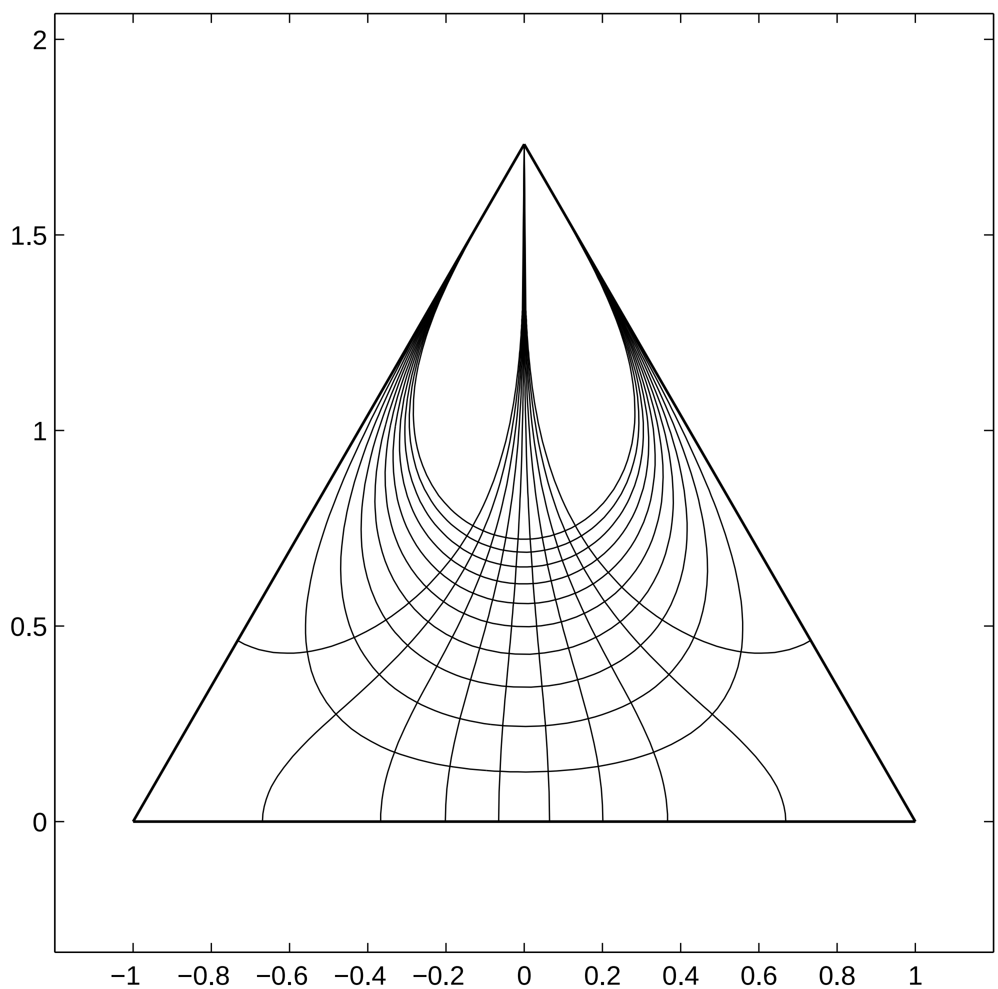
x_0
x_{i+1}
f
w_i
$f(z_0)$
w_{i+1}
θ
$f(z)$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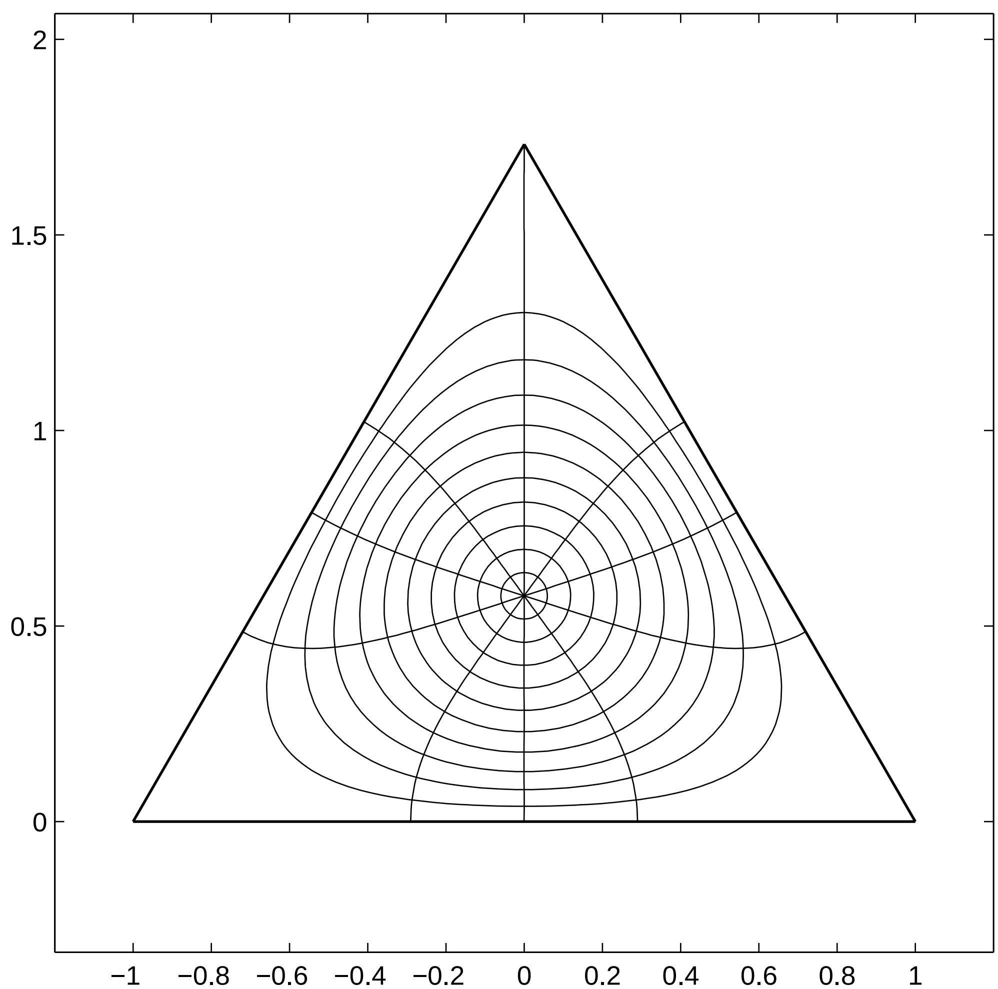
1
0.5
0
−1 −0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1


2
1.5
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0.8
1