Spinor

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In mathematics and physics, in particular in the theory of the orthogonal groups, spinors are elements of a complex vector space introduced to expand the notion of spatial vector. They are needed because the full structure of the group of rotations in a given number of dimensions requires some extra number of dimensions to exhibit it.

More formally, spinors can be defined as geometrical objects constructed from a given vector space endowed with a quadratic form by means of an algebraic\(^{[1]}\) or quantization\(^{[2]}\) procedure. A given quadratic form may support several different types of spinors. The ensemble of spinors of a given type is itself a vector space on which the rotation group acts, but for an ambiguity in the sign of the action. The space of spinors thus carries a projective representation of the rotation group. One can remove this sign ambiguity by regarding the space of spinors as a (linear) group representation of the spin group Spin(n). In this alternative point of view, many of the intrinsic and algebraic properties of spinors are more clearly visible, but the connection with the original spatial geometry is more obscure. On the other hand the use of complex number scalars can be kept to a minimum.

Spinors in general were discovered by Élie Cartan in 1913\(^{[3]}\). Later, spinors were adopted by quantum mechanics in order to study the properties of the intrinsic angular momentum of the electron and other fermions. Today spinors enjoy a wide range of physics applications. Classically, spinors in three dimensions are used to describe the spin of the non-relativistic electron. Via the Dirac equation, Dirac spinors are required in the mathematical description of the quantum state of the relativistic electron. In quantum field theory, spinors describe the state of relativistic many-particle systems.

In mathematics, particularly in differential geometry and global analysis, spinors have since found broad applications to algebraic and differential topology,\(^{[4]}\) symplectic geometry, gauge theory, complex algebraic geometry,\(^{[5]}\) index theory,\(^{[6]}\) and special holonomy.\(^{[7]}\)

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Overview

In the classical geometry of space, a vector exhibits a certain behavior when it is acted upon by a rotation or reflected in a hyperplane. However, in a certain sense rotations and reflections contain finer geometrical information than can be expressed in terms of their actions on vectors. Spinors are objects constructed in order to encompass more fully this geometry. (See orientation entanglement.)

There are essentially two frameworks for viewing the notion of a spinor.

One is representation theoretic. In this point of view, one knows a priori that there are some representations of the Lie algebra of the orthogonal group which cannot be formed by the usual tensor constructions. These missing representations are then labeled the spin representations, and their constituents spinors. In this view, a spinor must belong to a representation of the double cover of the rotation group SO(n,R), or more generally of the generalized special orthogonal group SO(p, q,R) on spaces with metric signature (p,q). These double-covers are Lie groups, called the spin groups Spin(p,q). All the properties of spinors, and their applications and derived objects, are manifested first in the spin group.

The other point of view is geometrical. One can explicitly construct the spinors, and then examine how they behave under the action of the relevant Lie groups. This latter approach has the advantage of providing a concrete and elementary description of what a spinor is. However, such a description becomes unwieldy when complicated properties of spinors, such as Fierz identities, are needed.

Clifford algebras

For more details on this topic, see Clifford algebra.

The language of Clifford algebras provides a complete picture of the spin representations of all the spin groups, and the various relationships between those representations, via the classification of Clifford algebras. It largely removes the need for ad hoc constructions, by introducing a type of geometric algebra.

Using the properties of Clifford algebras, it is then possible to determine the number and type of all irreducible spaces of spinors. In this view, a spinor is an element of the fundamental representation of the Clifford algebra \( \mathcal{C}_n(\mathbb{C}) \) over the complex numbers (or, more generally, of \( \mathcal{C}_{p,q}(\mathbb{R}) \) over the reals). In some cases it becomes clear that the spinors split into irreducible components under the action of Spin(p,q).

In detail, if \( V \) is a finite-dimensional complex vector space with nondegenerate bilinear form \( g \), the Clifford algebra is the algebra, \( \mathcal{C}(V, g) \), generated by \( V \) along with the anticommutation relation \( xy + yx = 2g(x,y) \). It is an abstract version of the algebra generated by the gamma matrices or Pauli matrices. The Clifford algebra \( \mathcal{C}_n(\mathbb{C}) \) is algebraically isomorphic to the algebra Mat(\( 2^k \),\( \mathbb{C} \)) of \( 2^k \times 2^k \) complex matrices, if \( n = \dim(V) = 2k \); or the algebra Mat(\( 2^k \),\( \mathbb{C} \)) opp Mat(\( 2^k \),\( \mathbb{C} \)) of two copies of the \( 2^k \times 2^k \) matrices, if \( n = \dim(V) = 2k+ 1 \). It therefore has a unique irreducible representation commonly denoted by \( \Delta \) of dimension \( 2^k \). Any such irreducible representation is, by definition, a space of spinors called a spin representation.

The subalgebra of the Clifford algebra spanned by products of an even number of vectors in \( V \)
contains the Lie algebra \( \mathfrak{so}(V,g) \) of the orthogonal group as a Lie subalgebra (under the commutator bracket). Consequently, \( \Delta \) is a representation of \( \mathfrak{so}(V,g) \). If \( n \) is odd, this representation is irreducible. If \( n \) is even, it splits again into two irreducible representations \( \Delta = \Delta_+ \oplus \Delta_- \) called the half-spin representations.

Irreducible representations in the case when \( V \) is a real vector space are much more intricate, and the reader is referred to the Clifford algebra article for more details.

**Terminology in physics**

The most typical type of spinor, the **Dirac spinor**,\(^9\) is an element of the fundamental representation of the complexified Clifford algebra \( \mathbb{C}\ell(p,q) \), into which the spin group \( \text{Spin}(p,q) \) may be embedded. On a \( 2k \)- or \( 2k+1 \)-dimensional space a Dirac spinor may be represented as a vector of \( 2^k \) complex numbers. (See Special unitary group.) In even dimensions, this representation is reducible when taken as a representation of \( \text{Spin}(p,q) \) and may be decomposed into two: the left-handed and right-handed **Weyl spinor**\(^{10}\) representations. In addition, sometimes the non-complexified version of \( \mathbb{C}\ell(p,q) \) has a smaller real representation, the **Majorana spinor** representation.\(^{11}\) If this happens in an even dimension, the Majorana spinor representation will sometimes decompose into two **Majorana-Weyl spinor** representations.

Of all these, only the Dirac representation exists in all dimensions. Dirac and Weyl spinors are complex representations while Majorana spinors are real representations.

**Spinors in representation theory**

One major mathematical application of the construction of spinors is to make possible the explicit construction of linear representations of the Lie algebras of the special orthogonal groups, and consequently spinor representations of the groups themselves. At a more profound level, spinors have been found to be at the heart of approaches to the index theorem, and to provide constructions in particular for discrete series representations of semisimple groups.

**History**

The most general mathematical form of spinors was discovered by Élie Cartan in 1913.\(^{12}\) The word "spinor" was coined by Paul Ehrenfest in his work on quantum physics.

Spinors were first applied to mathematical physics by Wolfgang Pauli in 1927, when he introduced spin matrices.\(^{13}\) The following year, Paul Dirac discovered the fully relativistic theory of electron spin by showing the connection between spinors and the Lorentz group.\(^{14}\) By the 1930s, Dirac, Piet Hein and others at the Niels Bohr Institute created games such as **Tangloids** to teach and model the calculus of spinors.

**Examples**

Some important simple examples of spinors in low dimensions arise from considering the even-graded subalgebras of the Clifford algebra \( \mathbb{C}\ell_{p,q}(\mathbb{R}) \). This is an algebra built up from an orthonormal basis of \( n = p + q \) mutually orthogonal vectors under addition and multiplication, \( p \) of which have norm +1 and \( q \) of which have norm −1, with the product rule for the basis vectors.
Two dimensions

The Clifford algebra $C_{2,0}(\mathbb{R})$ is built up from a basis of one unit scalar, 1, two orthogonal unit vectors, $\sigma_1$ and $\sigma_2$, and one unit pseudoscalar $i = \sigma_1 \sigma_2$. From the definitions above, it is evident that $(\sigma_1)^2 = (\sigma_2)^2 = 1$, and $(\sigma_1 \sigma_2)(\sigma_1 \sigma_2) = -\sigma_1 \sigma_2 \sigma_2 = -1$.

The even subalgebra $C_{0,2}(\mathbb{R})$, spanned by even-graded basis elements of $C_{2,0}(\mathbb{R})$, determines the space of spinors via its representations. It is made up of real linear combinations of 1 and $\sigma_1 \sigma_2$. As a real algebra, $C_{0,2}(\mathbb{R})$ is isomorphic to field of complex numbers $\mathbb{C}$. As a result, it admits a conjugation operation (analogous to complex conjugation), sometimes called the reverse of a Clifford element, defined by

$$(a + b \sigma_1 \sigma_2)^* = a + b \sigma_2 \sigma_1$$

which, by the Clifford relations, can be written

$$(a + b \sigma_1 \sigma_2)^* = a + b \sigma_2 \sigma_1 = a - b \sigma_1 \sigma_2$$

The action of an even Clifford element $\gamma \in C_{0,2}(\mathbb{R})$ on vectors, regarded as 1-graded elements of $C_{2,0}(\mathbb{R})$, is determined by mapping a general vector $u = a_1 \sigma_1 + a_2 \sigma_2$ to the vector

$$\gamma(u) = \gamma u \gamma^*,$$

where $\gamma^*$ is the conjugate of $\gamma$, and the product is Clifford multiplication. In this situation, a spinor [15] is an ordinary complex number. The action of $\gamma$ on a spinor $\phi$ is given by ordinary complex multiplication:

$$\gamma(\phi) = \gamma \phi.$$

An important feature of this definition is the distinction between ordinary vectors and spinors, manifested in how the even-graded elements act on each of them in different ways. In general, a quick check of the Clifford relations reveals that even-graded elements conjugate-commute with ordinary vectors:

$$\gamma(u) = \gamma u \gamma^* = \gamma^2 u.$$

On the other hand, comparing with the action on spinors $\gamma(\phi) = \gamma \phi$, $\gamma$ on ordinary vectors acts as the square of its action on spinors.

Consider, for example, the implication this has for plane rotations. Rotating a vector through an angle of $\theta$ corresponds to $\gamma^2 = \exp(\theta \sigma_1 \sigma_2)$, so that the corresponding action on spinors is via $\gamma = \pm \exp(\theta \sigma_1 \sigma_2/2)$. In general, because of logarithmic branching, it is impossible to choose a sign in a consistent way. Thus the representation of plane-rotations on spinors is two-valued.

In applications of spinors in two dimensions, it is common to exploit the fact that the algebra of even-graded elements (which is just the ring of complex numbers) is identical to the space of
spinors. So, by abuse of language, the two are often conflated. One may then talk about "the action of a spinor on a vector." In a general setting, such statements are meaningless. But in dimensions 2 and 3 (as applied, for example, to computer graphics) they make sense.

Examples

- The even-graded element

\[ \gamma = \frac{1}{\sqrt{2}}(1 - \sigma_1 \sigma_2) \]

corresponds to a vector rotation of 90° from \( \sigma_1 \) around towards \( \sigma_2 \), which can be checked by confirming that

\[ \frac{1}{2}(1 - \sigma_1 \sigma_2) \{a_1 \sigma_1 + a_2 \sigma_2\} (1 - \sigma_2 \sigma_1) = a_1 \sigma_2 - a_2 \sigma_1 \]

It corresponds to a spinor rotation of only 45°, however:

\[ \frac{1}{\sqrt{2}}(1 - \sigma_1 \sigma_2) \{a_1 + a_2 \sigma_1 \sigma_2\} = \frac{a_1 + a_2}{\sqrt{2}} + \frac{-a_1 + a_2}{\sqrt{2}} \sigma_1 \sigma_2 \]

- Similarly the even-graded element \( \gamma = -\sigma_1 \sigma_2 \) corresponds to a vector rotation of 180°:

\[ \{a_1 \sigma_1 + a_2 \sigma_2\} (\sigma_2 \sigma_1) = -a_1 \sigma_1 - a_2 \sigma_2 \]

but a spinor rotation of only 90°:

\[ \{a_1 + a_2 \sigma_1 \sigma_2\} = a_2 - a_1 \sigma_1 \sigma_2 \]

- Continuing on further, the even-graded element \( \gamma = -1 \) corresponds to a vector rotation of 360°:

\[ \{a_1 \sigma_1 + a_2 \sigma_2\} (-1) = a_1 \sigma_1 + a_2 \sigma_2 \]

but a spinor rotation of 180°.

Three dimensions

Main articles Spinors in three dimensions, Quaternions and spatial rotation

The Clifford algebra \( C_{3,0}(\mathbb{R}) \) is built up from a basis of one unit scalar, 1, three orthogonal unit vectors, \( \sigma_1, \sigma_2 \) and \( \sigma_3 \), the three unit bivectors \( \sigma_1 \sigma_2, \sigma_2 \sigma_3, \sigma_3 \sigma_1 \) and the pseudoscalar \( i = \sigma_1 \sigma_2 \sigma_3 \). It is straightforward to show that \( (\sigma_1)^2 = (\sigma_2)^2 = (\sigma_3)^2 = 1 \), and \( (\sigma_1 \sigma_2)^2 = (\sigma_2 \sigma_3)^2 = (\sigma_3 \sigma_1)^2 = (\sigma_1 \sigma_2 \sigma_3)^2 = -1 \).

The sub-algebra of even-graded elements is made up of scalar dilations,

\[ u' = \rho^{(1/2)} u \rho^{(1/2)} = \rho u, \]

and vector rotations

\[ u' = \gamma u \gamma^b, \]

where

\[
\gamma = \cos(\theta/2) - \left\{a_1 \sigma_2 \sigma_3 + a_2 \sigma_3 \sigma_1 + a_3 \sigma_1 \sigma_2\right\} \sin(\theta/2) \\
= \cos(\theta/2) - i\left\{a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3\right\} \sin(\theta/2) \\
= \cos(\theta/2) - iu \sin(\theta/2)
\]
corresponds to a vector rotation through an angle $\theta$ about an axis defined by a unit vector $v = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$.

As a special case, it is easy to see that if $v = \sigma_3$ this reproduces the $\sigma_1\sigma_2$ rotation considered in the previous section; and that such rotation leaves the coefficients of vectors in the $\sigma_3$ direction invariant, since

$$\left(\cos(\theta/2) - i\sigma_3\sin(\theta/2)\right)\sigma_3 \left(\cos(\theta/2) + i\sigma_3\sin(\theta/2)\right) = (\cos^2(\theta/2) + \sin^2(\theta/2)).$$

The bivectors $\sigma_2\sigma_3$, $\sigma_3\sigma_1$ and $\sigma_1\sigma_2$ are in fact Hamilton's quaternions $i$, $j$ and $k$, discovered in 1843:

$$i = -\sigma_2\sigma_3 = -i\sigma_1,$$
$$j = -\sigma_3\sigma_1 = -i\sigma_2,$$
$$k = -\sigma_1\sigma_2 = -i\sigma_3.$$

With the identification of the even-graded elements with the algebra $H$ of quaternions, as in the case of two-dimensions the only representation of the algebra of even-graded elements is on itself.\[16\]

Thus the (real\[17\]) spinors in three-dimensions are quaternions, and the action of an even-graded element on a spinor is given by ordinary quaternionic multiplication.

Note that the expression (1) for a vector rotation through an angle $\theta$, the angle appearing in $\gamma$ was halved. Thus the spinor rotation $\gamma(\psi) = \gamma\psi$ (ordinary quaternionic multiplication) will rotate the spinor $\psi$ through an angle one-half the measure of the angle of the corresponding vector rotation. Once again, the problem of lifting a vector rotation to a spinor rotation is two-valued: the expression (1) with $(180^\circ + \theta/2)$ in place of $\theta/2$ will produce the same vector rotation, but the negative of the spinor rotation.

The spinor/quaternion representation of rotations in 3D is becoming increasingly prevalent in computer geometry and other applications, because of the notable brevity of the corresponding spin matrix, and the simplicity with which they can be multiplied together to calculate the combined effect of successive rotations about different axes.

### Explicit constructions

A space of spinors can be constructed explicitly. For a complete example in dimension 3, see spinors in three dimensions. There are two different, but essentially equivalent, ways to proceed. One approach seeks to identify the minimal ideals for the left action of $Cl(V,g)$ on itself. These are subspaces of the Clifford algebra of the form $Cl(V,g)\omega$, admitting the evident action of $Cl(V,g)$ by left-multiplication: $c : x\omega \rightarrow cx\omega$. There are two variations on this theme: one can either find a primitive element $\omega$ which is a nilpotent element of the Clifford algebra, or one which is an idempotent. The construction via nilpotent elements is more fundamental in the sense that an idempotent may then be produced from it.\[18\] In this way, the spinor representations are identified with certain subspaces of the Clifford algebra itself. The second approach is to construct a vector space using a distinguished subspace of $V$, and then specify the action of the Clifford algebra externally to that vector space.

In either approach, the fundamental notion is that of an isotropic subspace $W$. Each construction depends on an initial freedom in choosing this subspace. In physical terms, this corresponds to the fact that there is no measurement protocol which can specify a basis of the spin space, even should a preferred basis of $V$ already be given.

As above, we let \((V, g)\) be an \(n\)-dimensional vector space equipped with a nondegenerate bilinear form. If \(V\) is a real vector space, then we replace \(V\) by its complexification \(\overline{V} \otimes_{\mathbb{R}} \mathbb{C}\) and let \(g\) denote the induced bilinear form on \(\overline{V} \otimes_{\mathbb{R}} \mathbb{C}\). Let \(W\) be a maximal subspace of \(V\) such that \(g|_W = 0\), (i.e., \(W\) is a maximal isotropic subspace). If \(n = 2k\) is even, then let \(W'\) be an isotropic subspace complementary to \(W\). If \(n = 2k+1\) is odd let \(W'\) be a maximal isotropic subspace with \(W \cap W' = 0\), and let \(U\) be the orthogonal complement of \(W \oplus W'\). In both the even and odd dimensional cases \(W\) and \(W'\) have dimension \(k\). In the odd dimensional case, \(U\) is one dimensional, spanned by a unit vector \(u\).

**Minimal ideals**

Since \(W\) is isotropic, multiplication of elements of \(W\) inside \(\mathcal{C}(V, g)\) is skew. Consequently, the \(k\)-fold product of \(W\) with itself, \(W^k\), is one-dimensional. Let \(\omega\) be a generator of \(W^k\). In terms of a basis of \(W\), \(w_1, \ldots, w_k\), one possibility is to set

\[
\omega = w_1 w_2 \ldots w_k.
\]

Note that \(\omega^2 = 0\) (i.e., \(\omega\) is nilpotent of order 2), and moreover, \(w_\omega = 0\) for all \(w \in W\). The following facts can be proven easily:

1. If \(n = 2k\), then the left ideal \(\Delta = \mathcal{C}(V, g)\omega\) is a minimal left ideal. Furthermore, this splits into the two spin spaces \(\Delta_+ = \mathcal{C}^{\text{even}}\omega\) and \(\Delta_- = \mathcal{C}^{\text{odd}}\omega\) on restriction to the action of the even Clifford algebra.
2. If \(n = 2k+1\), then the action of the unit vector \(u\) on the left ideal \(\mathcal{C}(V, g)\omega\) decomposes the space into a pair of isomorphic irreducible eigenspaces (both denoted by \(\Delta\)), corresponding to the respective eigenvalues \(+1\) and \(-1\).

In detail, suppose for instance that \(n\) is even. Suppose that \(I\) is a non-zero left ideal contained in \(\mathcal{C}(V, g)\omega\). We shall show that \(I\) must in fact be equal to \(\mathcal{C}(V, g)\omega\) by proving that it contains a nonzero scalar multiple of \(\omega\).

Fix a basis \(w_i\) of \(W\) and a complementary basis \(w_i'\) of \(W'\) so that

\[
w_i w_j' + w_j' w_i = \delta_{ij}, \quad \text{and} \quad (w_i')^2 = 0.
\]

Note that any element of \(I\) must have the form \(a\omega\), by virtue of our supposition that \(I \subseteq \mathcal{C}(V, g)\omega\). Let \(a\omega \in I\) be any such element. Using the chosen basis, we may write

\[
a = \sum_{i_1 < i_2 < \ldots < i_p} a_{i_1 \ldots i_p} w_{i_1}' \ldots w_{i_p}' + \sum_j B_j w_j
\]

where the \(a_{i_1 \ldots i_p}\) are scalars, and the \(B_j\) are auxiliary elements of the Clifford algebra. Pick any monomial \(a\) in this expansion of \(a\) having maximal homogeneous degree among the elements \(w_{i_p}'\):

\[
a = a_{i_1 \ldots i_p} w_{i_1}' \ldots w_{i_p}' \text{ (no summation implied)}
\]

Observe now that the product

\[
w_{i_p} \ldots w_{i_1} a \omega = a_{i_1 \ldots i_p} \omega
\]
is a nonzero scalar multiple of $\omega$, as required.

### Exterior algebra construction

Let $\Delta = \bigwedge W = \bigoplus_j \bigwedge^j W$ denote the exterior algebra of $W$ considered as vector space. This will be the spin representation, and its elements will be referred to as spinors.[19]

The action of the Clifford algebra on $\Delta$ is defined first by giving the action of an element of $V$ on $\Delta$, and then showing that this action respects the Clifford relation and so extends to a homomorphism of the full Clifford algebra into the endomorphism ring $\text{End}(\Delta)$ by the universal property of Clifford algebras. The details differ slightly according to whether the dimension of $V$ is even or odd.

When $\dim(V)$ is even, $V = W \oplus W'$ where $W'$ is the chosen isotropic complement. Hence any $v \in V$ decomposes uniquely as $v = w + w'$ with $w \in W$ and $w' \in W'$. The action of $v$ on a spinor is given by

$$c(v)w_1 \wedge \cdots \wedge w_n = (\varepsilon(w) + i(w'))(w_1 \wedge \cdots \wedge w_n)$$

where $i(w')$ is interior product with $w'$ using the non degenerate quadratic form to identify $V$ with $V^*$, and $\varepsilon(w)$ denotes the exterior product. It is easily verified that

$$c(u)c(v) + c(v)c(u) = 2g(u, v),$$

and so $c$ respects the Clifford relations and extends to a homomorphism from the Clifford algebra to $\text{End}(\Delta)$.

The spin representation $\Delta$ further decomposes into a pair of irreducible complex representations of the Spin group[20] (the half-spin representations, or Weyl spinors) via

$$\Delta_+ = \bigwedge^{\text{even}} W, \quad \Delta_- = \bigwedge^{\text{odd}} W.$$

When $\dim(V)$ is odd, $V = W \oplus U \oplus W'$, where $U$ is spanned by a unit vector $u$ orthogonal to $W$. The Clifford action $c$ is defined as before on $W \oplus W'$, while the Clifford action of (multiples of) $u$ is defined by

$$c(u)\alpha = \begin{cases} \alpha & \text{if } \alpha \in \bigwedge^{\text{even}} W \\ -\alpha & \text{if } \alpha \in \bigwedge^{\text{odd}} W \end{cases}$$

As before, one verifies that $c$ respects the Clifford relations, and so induces a homomorphism.

### Hermitian vector spaces and spinors

If the vector space $V$ has extra structure which provides a decomposition of its complexification into two maximal isotropic subspaces, then the definition of spinors (by either method) becomes natural.

The main example is the case that the real vector space $V$ is a hermitian vector space $(V, h)$, i.e., $V$ is equipped with a complex structure $J$ which is an orthogonal transformation with respect to the inner product $g$ on $V$. Then $V \otimes_{\mathbb{R}} \mathbb{C}$ splits in the $\pm J$eigenspaces of $J$. These eigenspaces are isotropic for the complexification of $g$ and can be identified with the complex vector space $(V, J)$ and its complex conjugate $(V, -J)$. Therefore for a hermitian vector space $(V, h)$ the vector space $\bigwedge V$ is a spinor...
space for the underlying real euclidean vector space.

With the Clifford action as above but with contraction using the hermitian form, this construction gives a spinor space at every point of an almost Hermitian manifold and is the reason why every almost complex manifold (in particular every symplectic manifold) has a \( \text{SpinC} \) structure. Likewise, every complex vector bundle on a manifold carries a \( \text{SpinC} \) structure.\(^{[21]}\)

**Clebsch-Gordan decomposition**

A number of Clebsch-Gordan decompositions are possible on the tensor product of one spin representation with another.\(^{[22]}\) These decompositions express the tensor product in terms of the alternating representations of the orthogonal group.

For the real or complex case, the alternating representations are

- \( \Gamma_r = \Lambda^r V \), the representation of the orthogonal group on skew tensors of rank \( r \).

In addition, for the real orthogonal groups, there are three characters (one-dimensional representations)

- \( \sigma_+ : O(p,q) \to \{-1,+1\} \) given by \( \sigma_+(R) = -1 \) if \( R \) reverses the spatial orientation of \( V \), \( +1 \) if \( R \) preserves the spatial orientation of \( V \). \((The \ spatial \ character.)\)
- \( \sigma_- : O(p,q) \to \{-1,+1\} \) given by \( \sigma_-(R) = -1 \) if \( R \) reverses the temporal orientation of \( V \), \( +1 \) if \( R \) preserves the temporal orientation of \( V \). \((The \ temporal \ character.)\)
- \( \sigma = \sigma_+ \sigma_- \). \((The \ orientation \ character.)\)

The Clebsch-Gordan decomposition allows one to define, among other things:

- An action of spinors on vectors.
- A Hermitian metric on the complex representations of the real spin groups.
- A Dirac operator on each spin representation.

**Even dimensions**

If \( n = 2k \) is even, then the tensor product of \( \Delta \) with the contragredient representation decomposes as

\[
\Delta \otimes \Delta^* \cong \bigoplus_{p=0}^{n} \Gamma_p \cong \bigoplus_{p=0}^{k-1} \left( \Gamma_p \oplus \sigma \Gamma_p \right) \oplus \Gamma_k
\]

which can be seen explicitly by considering (in the Explicit construction) the action of the Clifford algebra on decomposable elements \( \alpha \omega \square \beta \omega' \). The rightmost formulation follows from the transformation properties of the Hodge star operator. Note that on restriction to the even Clifford algebra, the paired summands \( \Gamma_p \oplus \sigma \Gamma_p \) are isomorphic, but under the full Clifford algebra they are not.

There is a natural identification of \( \Delta \) with its contragredient representation via the conjugation in the Clifford algebra:

\[
(\alpha \omega)^* = \omega (\alpha^*).
\]
So $\Delta \Delta$ also decomposes in the above manner. Furthermore, under the even Clifford algebra, the half-spin representations decompose

$$
\begin{align*}
\Delta_+ \otimes \Delta^*_+ & \cong \Delta_- \otimes \Delta^*_+ & \cong \bigoplus_{p=0}^{k} \Gamma_{2p} \\
\Delta_+ \otimes \Delta^*_+ & \cong \Delta_- \otimes \Delta^*_+ & \cong \bigoplus_{p=0}^{k+1} \Gamma_{2p+1}
\end{align*}
$$

For the complex representations of the real Clifford algebras, the associated reality structure on the complex Clifford algebra descends to the space of spinors (via the explicit construction in terms of minimal ideals, for instance). In this way, we obtain the complex conjugate $\Delta$ of the representation $\Delta$, and the following isomorphism is seen to hold:

$$\Delta \cong \sigma_- \Delta^*$$

In particular, note that the representation $\Delta$ of the orthochronous spin group is a unitary representation. In general, there are Clebsch-Gordan decompositions

$$\Delta \otimes \tilde{\Delta} \cong \bigoplus_{p=0}^{k} (\sigma_- \Gamma_p \oplus \sigma_+ \Gamma_p).$$

In metric signature $(p,q)$, the following isomorphisms hold for the conjugate half-spin representations

- If $q$ is even, then $\tilde{\Delta}_+ \cong \sigma_- \otimes \Delta^*_+$ and $\tilde{\Delta}_- \cong \sigma_- \otimes \Delta^*_+$.
- If $q$ is odd, then $\tilde{\Delta}_+ \cong \sigma_- \otimes \Delta^*_+$ and $\tilde{\Delta}_- \cong \sigma_- \otimes \Delta^*_+.$

Using these isomorphisms, one can deduce analogous decompositions for the tensor products of the half-spin representations $\Delta_+ \otimes \tilde{\Delta}_\pm$.

### Odd dimensions

If $n = 2k+1$ is odd, then

$$\Delta \otimes \Delta^* \cong \bigoplus_{p=0}^{k} \Gamma_{2p}.$$

In the real case, once again the isomorphism holds

$$\Delta \cong \sigma_- \Delta^*.$$

Hence there is a Clebsch-Gordan decomposition (again using the Hodge star to dualize) given by

$$\Delta \otimes \tilde{\Delta} \cong \sigma_- \Gamma_0 \oplus \sigma_+ \Gamma_1 \oplus \ldots \oplus \sigma_\pm \Gamma_k$$

### Consequences

There are many far-reaching consequences of the Clebsch-Gordan decompositions of the spinor
spaces. The most fundamental of these pertain to Dirac's theory of the electron, among whose basic requirements are

- A manner of regarding the product of two spinors as a scalar. In physical terms, a spinor should determine a probability amplitude for the quantum state.
- A manner of regarding the product as a vector. This is an essential feature of Dirac's theory, which ties the spinor formalism to the geometry of physical space.
- A manner of regarding a spinor as acting upon a vector, by an expression such as \( \psi \Delta \bar{\psi} \). In physical terms, this represents an electrical current of Maxwell's electromagnetic theory, or more generally a probability current.

**Summary in low dimensions**

- In 1 dimension (a trivial example), the single spinor representation is formally Majorana, a real 1-dimensional representation that does not transform.
- In 2 Euclidean dimensions, the left-handed and the right-handed Weyl spinor are 1-component complex representations, i.e. complex numbers that get multiplied by \( e^{\pm i\phi/2} \) under a rotation by angle \( \phi \).
- In 3 Euclidean dimensions, the single spinor representation is 2-dimensional and quaternionic. The existence of spinors in 3 dimensions follows from the isomorphism of the groups \( SU(2) \cong Spin(3) \) which allows us to define the action of \( Spin(3) \) on a complex 2-component column (a spinor); the generators of \( SU(2) \) can be written as Pauli matrices.
- In 4 Euclidean dimensions, the corresponding isomorphism is \( Spin(4) \cong SU(2) \times SU(2) \). There are two inequivalent quaternionic 2-component Weyl spinors and each of them transforms under one of the \( SU(2) \) factors only.
- In 5 Euclidean dimensions, the relevant isomorphism is \( Spin(5) \cong USp(4) \cong Sp(2) \) which implies that the single spinor representation is 4-dimensional and quaternionic.
- In 6 Euclidean dimensions, the isomorphism \( Spin(6) \cong SU(4) \) guarantees that there are two 4-dimensional complex Weyl representations that are complex conjugates of one another.
- In 7 Euclidean dimensions, the single spinor representation is 8-dimensional and real; no isomorphisms to a Lie algebra from another series (A or C) exist from this dimension on.
- In 8 Euclidean dimensions, there are two Weyl-Majorana real 8-dimensional representations that are related to the 8-dimensional real vector representation by a special property of \( Spin(8) \) called triality.
- In \( d + 8 \) dimensions, the number of distinct irreducible spinor representations and their reality (whether they are real, pseudoreal, or complex) mimics the structure in \( d \) dimensions, but their dimensions are 16 times larger; this allows one to understand all remaining cases. See Bott periodicity.
- In spacetimes with \( p \) spatial and \( q \) time-like directions, the dimensions viewed as dimensions over the complex numbers coincide with the case of the \( p + q \)-dimensional Euclidean space, but the reality projections mimic the structure in \( |p - q| \) Euclidean dimensions. For example, in 3+1 dimensions there are two non-equivalent Weyl complex (like in 2 dimensions) 2-component (like in 4 dimensions) spinors, which follows from the isomorphism.
$SL(2,C) \equiv Spin(3,1)$

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<th>Dirac</th>
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See also

- Spinor bundle
- Pure spinor
- Anyon
- Twistor
- Plate trick
- Supercharge
- Dirac equation in the algebra of physical space

Notes

1. ^ The conventional algebraic route to their discussion is through the theory of Clifford algebras, which produce naturally the basic spin representation.
2. ^ Another approach, which at one time had its heyday, but now has waned in popularity, is to construct the Clifford algebra *ex nihilo* as a matrix algebra by "quantizing" the coordinates in the original vector space. From this framework, spinors are simply the column vectors on which the matrices act. One may then appeal to techniques from linear algebra directly to split the spaces of spinors into irreducible parts.
3. ^ Cartan 1913.
7. ^ Lawson & Michelsohn 1989, Harvey 1990. These two books also provide good mathematical introductions and fairly comprehensive bibliographies on the mathematical applications of spinors as of

12. ^ Cartan 1913
15. ^ These are the right-handed Weyl spinors in two-dimensions. For the left-handed Weyl spinors, the representation is via $\gamma(\phi) = \tilde{\gamma}\phi$. The Majorana spinors are the common underlying real representation for the Weyl representations.
16. ^ Since, for a skew field, the kernel of the representation must be trivial. So inequivalent representations can only arise via an automorphism of the skew-field. In this case, there are a pair of equivalent representations: $\gamma(\phi) = \gamma_0$, and its quaternionic conjugate $\gamma(\phi) = \phi\tilde{\gamma}$.
17. ^ The complex spinors are obtained as the representations of the tensor product $H \otimes_{\mathbb{R}} \mathbb{C} = \text{Mat}_2(\mathbb{C})$. These are considered in more detail in spinors in three dimensions.
18. ^ This construction is due to Cartan. The treatment here is based on Chevalley (1954).
19. ^ One source for this subsection is Fulton & Harris (1991).
20. ^ Via the even-graded Clifford algebra.
21. ^ Lawson & Michelsohn 1989, Appendix D.

References


Categories: Spinors | Rotational symmetry | Quantum field theory
Hidden categories: All articles with unsourced statements | Articles with unsourced statements since September 2008