

Z-transform

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In mathematics and signal processing, the **Z-transform** converts a discrete time domain signal, which is a sequence of real numbers, into a complex frequency domain representation.

The Z-transform and advanced Z-transform were introduced (under the Z-transform name) by E. I. Jury in 1958 in *Sampled-Data Control Systems* (John Wiley & Sons). The idea contained within the Z-transform was previously known as the "generating function method".

The (unilateral) Z-transform is to discrete time domain signals what the one-sided Laplace transform is to continuous time domain signals.

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Definition

The Z-transform, like many other integral transforms, can be defined as either a *one-sided* or *two-sided* transform.

Bilateral Z-Transform

The *bilateral* or *two-sided* Z-transform of a discrete-time signal $x[n]$ is the function $X(z)$ defined as

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where n is an integer and z is, in general, a complex number:

$$z = Ae^{j\varphi}$$

where A is the magnitude of z , and φ is the angular frequency (in radians per sample).

Unilateral Z-Transform

Alternatively, in cases where $x[n]$ is defined only for $n \geq 0$, the *single-sided* or *unilateral* Z-transform is defined as

$$X(z) = Z\{x[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n}$$

In signal processing, this definition is used when the signal is causal.

An important example of the unilateral Z-transform is the probability-generating function, where the component $x[n]$ is the probability that a discrete random variable takes the value n , and the function $X(z)$ is usually written as $X(s)$, in terms of $s = z^{-1}$. The properties of Z-transforms (below) have useful interpretations in the context of probability theory.

Inverse Z-Transform

The **inverse Z-Transform** is

$$x[n] = Z^{-1}\{X(z)\} = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} dz$$

where C is a counterclockwise closed path encircling the origin and entirely in the region of convergence (ROC). The contour or path, C , must encircle all of the poles of $X(z)$.

A special case of this contour integral occurs when C is the unit circle (and can be used when the ROC includes the unit circle). The inverse Z-Transform simplifies to the **inverse Discrete-Time Fourier transform**:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n} d\omega.$$

The Z-transform with a finite range of n and a finite number of uniformly-spaced z values can be computed efficiently via Bluestein's FFT algorithm. The discrete time Fourier transform (DTFT) (not to confuse with the discrete Fourier transform (DFT)) is a special case of such a Z-transform obtained by restricting z to lie on the unit circle.

Region of convergence

The region of convergence (ROC) is where the Z-transform of a signal has a finite sum for a region in the complex plane.

$$ROC = \left\{ z : \sum_{n=-\infty}^{\infty} x[n]z^{-n} < \infty \right\}$$

Example 1 (No ROC)

Let $x[n] = 0.5^n$. Expanding $x[n]$ on the interval $(-\infty, \infty)$ it becomes

$$x[n] = \{\dots, 0.5^{-3}, 0.5^{-2}, 0.5^{-1}, 1, 0.5, 0.5^2, 0.5^3, \dots\} = \{\dots, 2^3, 2^2, 2, 1, 0.5, 0.5^2, 0.5^3, \dots\}$$

Looking at the sum

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n} < \infty$$

There are no such values of z that satisfy this condition.

Example 2 (causal ROC)

Let $x[n] = 0.5^n u[n]$ (where u is the Heaviside step function). Expanding $x[n]$ on the interval $(-\infty, \infty)$ it becomes

$$x[n] = \{ \dots, 0, 0, 0, 1, 0.5, 0.5^2, 0.5^3, \dots \}$$

Looking at the sum

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} 0.5^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^n = \frac{1}{1 - 0.5z^{-1}}$$

The last equality arises from the infinite geometric series and the equality only holds if $\left|0.5z^{-1}\right| < 1$ which can be rewritten in terms of z as $|z| > 0.5$. Thus, the ROC is $|z| > 0.5$. In this case the ROC is the complex plane with a disc of radius 0.5 at the origin "punched out".

Example 3 (anticausal ROC)

Let $x[n] = -(0.5)^n u[-n - 1]$ (where u is the Heaviside step function). Expanding $x[n]$ on the interval $(-\infty, \infty)$ it becomes

$$x[n] = \{ \dots, -(0.5)^{-3}, -(0.5)^{-2}, -(0.5)^{-1}, 0, 0, 0, \dots \}$$

Looking at the sum

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x[n]z^{-n} &= - \sum_{n=-\infty}^{-1} 0.5^n z^{-n} = - \sum_{n=-\infty}^{-1} \left(\frac{z}{0.5}\right)^{-n} \\ &= - \sum_{m=1}^{\infty} \left(\frac{z}{0.5}\right)^m = - \frac{0.5^{-1}z}{1 - 0.5^{-1}z} = \frac{z}{z - 0.5} = \frac{1}{1 - 0.5z^{-1}} \end{aligned}$$

Using the infinite geometric series, again, the equality only holds if $\left|0.5^{-1}z\right| < 1$ which can be rewritten in terms of z as $|z| < 0.5$. Thus, the ROC is $|z| < 0.5$. In this case the ROC is a disc centered at the origin and of radius 0.5.

Examples conclusion

Examples 2 & 3 clearly show that the Z-transform $X(z)$ of $x[n]$ is unique when and only when specifying the ROC. Creating the pole-zero plot for the causal and anticausal case show that the ROC for either case does not include the pole that is at 0.5. This extends to cases with multiple poles: the ROC will **never** contain poles.

In example 2, the causal system yields an ROC that includes $|z| = \infty$ while the anticausal system in example 3 yields an ROC that includes $|z| = 0$.

In systems with multiple poles it is possible to have an ROC that includes neither $|z| = \infty$ nor $|z| = 0$. The ROC creates a circular band. For example, $x[n] = 0.5^n u[n] - 0.75^n u[-n - 1]$ has poles at 0.5 and 0.75. The ROC will be $0.5 < |z| < 0.75$, which includes neither the origin nor infinity. Such a system is called a mixed-causality system as it contains a causal term $0.5^n u[n]$ and an anticausal term $-(0.75)^n u[-n - 1]$.

The stability of a system can also be determined by knowing the ROC alone. If the ROC contains the unit circle (i.e., $|z| = 1$) then the system is stable. In the above systems the causal system is stable because $|z| > 0.5$ contains the unit circle.

If you are provided a Z-transform of a system without an ROC (i.e., an ambiguous $x[n]$) you can determine a unique $x[n]$ provided you desire the following:

- Stability
- Causality

If you need stability then the ROC must contain the unit circle. If you need a causal system then the ROC must contain infinity. If you need an anticausal system then the ROC must contain the origin.

The unique $x[n]$ can then be found.

Properties

Properties of the z-transform

	Time domain	Z-domain	ROC
Notation	$x[n] = \mathcal{Z}^{-1}\{X(z)\}$	$X(z) = \mathcal{Z}\{x[n]\}$	ROC: $r_2 < z < r_1$
Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC ₁ and ROC ₂
Time shifting	$x[n - k]$	$z^{-k}X(z)$	ROC, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x[n]$	$X(a^{-1}z)$	$ a r_2 < z < a r_1$
Time reversal	$x[-n]$	$X(z^{-1})$	$\frac{1}{r_2} < z < \frac{1}{r_1}$
Conjugation	$x^*[n]$	$X^*(z^*)$	ROC
Real part	$\text{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	ROC
Imaginary part	$\text{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	ROC
Differentiation	$nx[n]$	$-z \frac{dX(z)}{dz}$	ROC
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of ROC ₁ and ROC ₂
Correlation	$r_{x_1,x_2}(l) = x_1[l] * x_2[-l]$	$R_{x_1,x_2}(z) = X_1(z)X_2(z^{-1})$	At least the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Multiplication	$x_1[n]x_2[n]$	$\frac{1}{j2\pi} \oint_C X_1(v)X_2(\frac{z}{v})v^{-1}dv$	At least $r_{1l}r_{2l} < z < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]$	$\frac{1}{j2\pi} \oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}dv$	

■ **Initial value theorem**

$$x[0] = \lim_{z \rightarrow \infty} X(z), \text{ If } x[n] \text{ causal}$$

■ **Final value theorem**

$$x[\infty] = \lim_{z \rightarrow 1} (1 - z^{-1})X(z), \text{ Only if poles of } (z - 1)X(z) \text{ are inside the unit circle}$$

Table of common Z-transform pairs

Signal, $x[n]$	Z-transform, $X(z)$	ROC
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1	$\delta[n]$	1	all z
2	$\delta[n - n_0]$	$\frac{1}{z^{n_0}}$	$ z > 0$
3	$u[n]$	$\frac{z}{z - 1}$	$ z > 1$
4	$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z > a $
5	$na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z > a $
6	$-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z < a $
7	$-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z < a $
8	$\cos(\omega_0 n) u[n]$	$\frac{1 - z^{-1} \cos(\omega_0)}{1 - 2z^{-1} \cos(\omega_0) + z^{-2}}$	$ z > 1$
9	$\sin(\omega_0 n) u[n]$	$\frac{z^{-1} \sin(\omega_0)}{1 - 2z^{-1} \cos(\omega_0) + z^{-2}}$	$ z > 1$
10	$a^n \cos(\omega_0 n) u[n]$	$\frac{1 - az^{-1} \cos(\omega_0)}{1 - 2az^{-1} \cos(\omega_0) + a^2 z^{-2}}$	$ z > a $
11	$a^n \sin(\omega_0 n) u[n]$	$\frac{az^{-1} \sin(\omega_0)}{1 - 2az^{-1} \cos(\omega_0) + a^2 z^{-2}}$	$ z > a $

Relationship to Laplace

The bilateral Z-transform is simply the two-sided Laplace transform of the ideal sampled function

$$x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT)$$

where $x(t)$ is the continuous-time function being sampled, $x[n] = x(nT)$ the n^{th} sample, T is the sampling period, and with the substitution: $z = e^{sT}$.

Likewise the unilateral Z-transform is simply the one-sided Laplace transform of the ideal sampled function. Both assume that the sampled function is zero for all negative time indices.

The Bilinear transform is a useful approximation for converting continuous time filters (represented in Laplace space) into discrete time filters (represented in z space), and vice versa. To do this, you can use the following substitutions in $H(s)$ or $H(z)$:

$$s = \frac{2}{T} \frac{z - 1}{z + 1} \text{ from Laplace to } z \text{ (Tustin transformation);}$$

$$z = \frac{2 + sT}{2 - sT} \text{ from } z \text{ to Laplace.}$$

Relationship to Fourier

The Z-transform is a generalization of the discrete time fourier transform (DTFT). The DTFT can be found by evaluating the Z-transform $X(z)$ at $z = e^{j\omega}$ or, in other words, evaluated on the unit circle. In order to determine the frequency response of the system the Z-transform must be evaluated on the unit circle, meaning that the system's region of convergence must contain the unit circle. Otherwise, the DTFT of the system does not exist.

Linear constant coefficient difference equation

The linear constant coefficient difference (LCCD) equation is a representation for a linear system based on the autoregressive moving average equation.

$$\sum_{p=0}^N y[n-p]\alpha_p = \sum_{q=0}^M x[n-q]\beta_q$$

Both sides of the above equation can be divided by α_0 , if it is not zero, normalizing $\alpha_0 = 1$ and the LCCD equation can be written

$$y[n] = \sum_{q=0}^M x[n-q]\beta_q - \sum_{p=1}^N y[n-p]\alpha_p$$

This form of the LCCD equation is favorable to make it more explicit that the "current" output $y[n]$ is a function of past outputs $y[n-p]$, current input $x[n]$, and previous inputs $x[n-q]$.

Transfer function

Taking the Z-transform of the equation yields

$$Y(z) \sum_{p=0}^N z^{-p}\alpha_p = X(z) \sum_{q=0}^M z^{-q}\beta_q$$

and rearranging results in

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{q=0}^M z^{-q}\beta_q}{\sum_{p=0}^N z^{-p}\alpha_p} = \frac{\beta_0 + z^{-1}\beta_1 + z^{-2}\beta_2 + \dots + z^{-M}\beta_M}{\alpha_0 + z^{-1}\alpha_1 + z^{-2}\alpha_2 + \dots + z^{-N}\alpha_N}$$

Zeros and poles

From the fundamental theorem of algebra the numerator has M roots (called zeros) and the denominator has N roots (called poles). Rewriting the transfer function in terms of poles and zeros

$$H(z) = \frac{(1 - q_1z^{-1})(1 - q_2z^{-1})\dots(1 - q_Mz^{-1})}{(1 - p_1z^{-1})(1 - p_2z^{-1})\dots(1 - p_Nz^{-1})}$$

Where q_k is the k^{th} zero and p_k is the k^{th} pole. The zeros and poles are commonly complex and when plotted on the complex plane (z-plane) it is called the pole-zero plot.

In simple words, zeros are the solutions to the equation obtained by setting the numerator equal to zero, while poles are the solutions to the equation obtained by setting the denominator equal to zero.

In addition, there may also exist zeros and poles at $z = 0$ and $z = \infty$. If we take these poles and zeros as well as multiple-order zeros and poles into consideration, the number of zeros and poles are always equal.

By factoring the denominator, partial fraction decomposition can be used, which can then be transformed back to the time domain. Doing so would result in the impulse response and the linear constant coefficient difference equation of the system.

Output response

If such a system $H(z)$ is driven by a signal $X(z)$ then the output is $Y(z) = H(z)X(z)$. By performing partial fraction decomposition on $Y(z)$ and then taking the inverse Z-transform the output $y[n]$ can be found. In practice, it is often useful to fractionally decompose $\frac{Y(z)}{z}$ before multiplying that quantity by z to generate a form of $Y(z)$ which has terms with easily computable inverse Z-transforms.

See also

- Zeta function regularization
- Advanced Z-transform
- Formal power series
- Laplace transform

Bibliography

- Eliahu Ibrahim Jury, *Theory and Application of the Z-Transform Method*, Krieger Pub Co, 1973. ISBN 0-88275-122-0.
- Refaat El Attar, *Lecture notes on Z-Transform*, Lulu Press, Morrisville NC, 2005. ISBN 1-4116-1979-X.

External links

- Mathworld's entry on the Z-transform (<http://mathworld.wolfram.com/Z-Transform.html>)
- Z-Transform threads in Comp.DSP (http://www.dsprelated.com/comp.dsp/keyword/Z_Transform.php)
- Z-Transform Module by John H. Mathews (<http://math.fullerton.edu/mathews/c2003/ZTransformIntroMod.html>)

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